UNIT V

Multivariate Signal Processing

A Multivariate Time Series (MTS) is a collection of multiple time-dependent variablesobserved simultaneously over time. Unlike a univariate time series (which tracksone variable over time, like daily temperature), a multivariate time series records two or more related variables, capturing their interdependencies andjoint

Multivariate Time Series

dynamics.

Definition

- Suppose we have p variables observed across T time steps.
- A multivariate time series can be represented as:

$$X_t = (x_{1t}, x_{2t}, \ldots, x_{pt}), \quad t = 1, 2, \ldots, T$$

where each x_{it} is the value of variable i at time t.

Examples
– Weather forecasting: temperature, humidity, wind speed recorded daily. – Finance: stock price, trading volume, interest rates over time. – Healthcare: heart rate, blood pressure, oxygen level measured over time.

Random Processes / Random Variables

A **random process** $\{X(t)\}_{t\in T}$ is a collection of RVs indexed by time t:

$$X(t):\Omega\to\mathbb{R},\quad t\in T,$$

where T is the index set (discrete $T=\mathbb{Z}$ or continuous $T=\mathbb{R}$).

First-order (mean) function

Mean function:

$$m_X(t)=\mathbb{E}[X(t)].$$

If $m_X(t)=\mu$ (constant) for all t, the mean is time-invariant (important in stationary processes).

Second-order functions

Autocorrelation (raw second moment):

$$R_X(t_1,t_2)=\mathbb{E}\big[X(t_1)X(t_2)\big].$$

Autocovariance (central):

$$C_X(t_1,t_2)=\mathbb{E}\big[(X(t_1)-m_X(t_1))(X(t_2)-m_X(t_2))\big].$$

Relationship:

$$R_X(t_1,t_2)=C_X(t_1,t_2)+m_X(t_1)m_X(t_2).$$

Marginal (single-time) CDF / PDF at time t:

$$F_{X(t)}(x) = \mathbb{P}(X(t) \le x), \qquad f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x)$$

For any finite set of times $t_1, \ldots, t_n$, the **joint cumulative distribution function (joint CDF)** is

$$F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n) = \mathbb{P}\big(X(t_1) \le x_1, \ldots, X(t_n) \le x_n\big).$$

$$f_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n).$$

Strict-sense stationarity (SSS) — definition

A stochastic process $\{X(t)\}_{t \in T}$ is **strict-sense stationary** if **all** its finite-dimensional distributions are invariant under time shifts.
Formally: for every n, every times $t_1, \ldots, t_n$ and every shift τ,

$$F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n) = F_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1, \ldots, x_n).$$

Wide-sense (second-order) stationarity (WSS) — definition

If second moments exist, $\{X(t)\}$ is **wide-sense stationary (WSS)** if:

1. the mean is constant (time-invariant)

$$m_X(t) = \mathbb{E}[X(t)] = \mu \quad \text{for all } t,$$

2. the autocovariance depends only on the lag τ (not on absolute time)

$$C_X(t_1, t_2) = \mathrm{Cov}(X(t_1), X(t_2)) = C_X(\tau), \qquad \tau = t_2 - t_1.$$

In discrete time use $X[n]$, mean $m_X[n] = \mu$ and $C_X[k, l] = C_X[l - k]$.

- Hermitian symmetry

Properties of WSS

Statement. For real processes $R_X(\tau) = R_X(-\tau)$. For complex-valued processes $R_X(\tau) = R_X^*(-\tau)$.

Proof.

$$R_X(\tau) = \mathbb{E}[X[n]X[n+\tau]].$$

Take complex conjugate:

$$R_X(\tau)^* = \mathbb{E}[X[n]X[n+\tau]]^* = \mathbb{E}[X[n+\tau]^* X[n]^*] = \mathbb{E}[X^*[n+\tau]\, X^*[n]] = R_{X^*}(-\tau).$$

•

Boundedness:

- Positive semidefiniteness

- Power spectral density (PSD) exists as DTFT of autocorrelation (Wiener–Khinchin)

Proof:

Autocorrelation

Properties

Spectral Factorization

• Ergodicity refers to time averages (along one long realization) • Ergodic in mean:

A process is ergodic in mean if the average computed by

Ergodicity

= ensemble averages (over many realizations). • In practice, it is the property that allows to estimate statistical quantities (mean, autocovariance, PSD) from a single long recorded signal instead of needing many independent realizations.

sliding along one long realization (one long recording) converges to the true average if observes over many independent realizations. In other words:

• Time average = Ensemble average (for the mean).

Time Series Models

• 1. Auto Regressive or (AR Model). • 2. Moving Average, or (MA model). • 3. Autoregressive and Moving Average, or

(ARMA model).

AR Model (All Pole Model)

Yule–Walker equations (derive AR coefficients from autocovariances)

- PSD of AR(p)

ARMA (Pole-Zero) Model

- EXAMPLE

The estimated autocorrelation sequence of a random process x(n) for lags k=0,1, 2, 3, 4 are rx(0)=2, rx(1)=1, rx(2)=1, rx(3)=0.5, rx(4)=0 Compute the power spectrum of x(n) for each of the following cases. (a) x(n) is an AR(2) process.

(b) x(n) is an ARMA(1,1) process.

- Given